PURE MATHEMATICS AND MODERN SCIENCE[1]

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RESUMEN

Busco dar una visión de las matemáticas puras como parte de una visión general sobre la naturaleza de las matemáticas. Se delinean dos interpretaciones sobre la evolución de la ciencia moderna y las matemáticas, aunque ambas sirven para captar el sentido de esa realidad. El artículo asume una ontología y una epistemología de corte empirista, aunque solo hasta cierto punto.

Mathematics starts with an intellectual practice on concepts derived from Mathematical objects. The simple use of operations should not carry away Mathematics "very far" from its object. I suggest that when new representations are introduced in the practice, or when the operations are deconstructed and reconstructed in a qualitatively different form, then the process to carry away further starts. Let me try to explain this better.

The Mathematics practice is conducted with certain operations and representations. The range of abstraction of this practice changes -we can say there are "level jumps"- through changes in the operations and in the representations used. These "level-jumps" can be tracked historically. This implies concrete historical analysis.[2] It is worthy to note that the change in representations might correspond to the introduction of new objects or not (in the latter case obeying strictly mental constructs).

The possibility of the use of these changes of representations and operations, and operation qualities, was developed in Mathematics because of the special types of objects it has. This can be tracked -again- historically. In my opinion, this type of intellectual processes can be tracked in other sciences as well. But it is clear that Mathematics became a practice specialized in this type of

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[1] This paper was written as a result of the research I developed in the Department of the History of Science of Harvard University during 1989. I am very grateful to the Council of International Exchange of Scholars of the USA for the Fulbright Grant I received to conduct my research.

[2] Even if I underline the historical "input" of history in the evolution of mathematics, I do not agree with historicist views as the one Spengler developed.

processes. [3] The XIX th century was the birth place of this substantial change in the history of the mathematical practice.

The former analysis brings us to an interesting point. The description of the objects as it was before the outcome of a new practice which emerges in the XIXth century, until a certain point was abandoned for a new form of practice. The question is whether the changes of the XIXth century changed at the same time the character of Mathematics? That would be to say that Mathematics got out of the description of a world of empirical objects. My answer is no. I suggest that there is interplay between two worlds: between the world of Mathematics objects and the world of Mathematics concepts and operations. But a change in the type of these relations occurred. Before it was possible to follow through a "link to perception" the consequences of Mathematics practice in the vast majority of the Mathematics building. When the revolution in Mathematics occurred this was not possible to the same extent anymore. [4]

What this means is that the new Mathematics -more than ever- can have description of objects and no-description of objects. Some parts of Mathematics do not describe the objects. [5]

On the other hand, the revolution in Mathematics brought also the approach to different intermingled Mathematics object-realms. The revolution was bringing together -in precise forms- different realms and different methods. Later, we will get into a more profound analysis of the role of methods in Mathematics.

We are going to say something about an issue we will analyze again later. If there are parts of mathematics that are descriptions of an objects-world and others that are not, what would be criteria to make a rational distinction between both? Or is such a question epistemologically valid? This question is connected with the problem of truth in Mathematics. [6]

I would suggest that -without getting into the solution of this riddle- we can use the analysis of the representations and abstractions done. This gives criteria to gather information in the strategy to determine when Mathematics theories can be describing or not the world of Mathematics objects. This

[3] Normally, historical accounts show more about the history of methods than about objects.

See Boyer 1968, Bourbaki 1974, etc.

[4] See Kline 1980, for an appropriate interpretation of the meaning of abstraction in XIXth Century Mathematics.

[5] See Ruiz-Zúñiga 1987 (c).

[6] For an excellent collection of essays on these issues consult Benacerraf and Putnam 1964.

criterion implies the use of historical analysis and theoretical studies. But these criteria are not enough to sanction the description or non description.

The former paragraph brings up the problem of the status of truth in Mathematics. I was suggesting a method which involves essentially theoretical analysis and not confrontation with the world in the way other sciences do. The issue here is if Mathematics needs that confrontation or not? Or, what is the empirical confrontation Mathematics should perform in order to establish its truth? Where do the differences with other sciences arise from?

Perhaps if we start from the latter we can answer all the questions raised. We can stress out the difference between Mathematics objects and other science objects in terms of the quality and length of the collection of perceptions Mathematics supposes. But, we can add something else to that. Mathematics objects differ in another way to other science objects. [7] The objects of Mathematics are generally founded. Somehow, space-diversity or space-shapes or time-sequentiality can be found in all space or time realities; infinite processes, continuity and discontinuity, necessity and randomness are present in everything. Mathematics objects can be described as members of a "selected" club: the club of the "general". This brings up the issue of whether Mathematics objects are only the abstractions. I have underlined a cognitive structure where there are abstractions and perceptions and external objects. This applies to all particular sciences.

Now, I am coming back to objects again.[8] There is a qualitative difference between Mathematics objects. I do not find that a mathematical consequence from a triangle can lead to "exact" result in the empirical world. But in arithmetic it does. This suggests the different epistemological status for different objects and parts of Mathematics. This issue should be understood in a global form. The so-called "exactness" of Mathematics is many times misunderstood. The correspondence of certain scientific theories with reality is not the same that other theories may have. There is a wide diversity of approximation or correspondence to reality within science. The same thing occurs in Mathematics. The correspondence with reality depends on the object and on the methods used. Thus, there is not that exactness of Mathematics if this is understood as correspondence to reality. On the other hand, if "exactness" refers to the world of concepts and operations, this is feasible as a

[7] First of all, I do believe in the existence of a close relationship between the objects of all sciences; the differences I want to state here are more for epistemological convenience.

[8] It is clear that I do not believe in Platonism. Classical Platonist approaches can be found in Frege 1950 and Gödel 1981. An assessment of Platonism can be studied in Hawkins 1985.

way of description of the derivation of consequences within a theory or a field where there are clearly defined rules, operations, and basic concepts and propositions. [9]

Research should give us an orderly structure of Mathematics objects. [10] It should be possible to determine the precise epistemological terms through which Mathematics objects can be described. [11]

However, in both cases there is a description of perception or perceptible objects. [12]

I may suggest that the dimension for description of the real we defined has been almost lost as a definition of Mathematical practice, especially with the emergence of pure Mathematics. [13] The emphasis has been given to the creation and abstraction. This has driven Mathematics towards the building of a huge part of results without connection to reality and useless in that sense, but useful in another sense. This situation has contributed meaningfully to the reconstruction of science in the modern times (since the XVIIth century). We will come back to this.

On the other hand, the general nature of Mathematics objects creates the possibility of wider uses of Mathematics results. We can talk about "more applications". Nevertheless, this depends on the part of Mathematics we are talking about. It is not the same for Geometry as for Arithmetic, as we stated before. [14]

The issue of Mathematics methods was introduced before. Now, I should advance my analysis. There is a separation between methods and objects. What I want to stress here is that methods can be applied to different objects, not necessarily the objects to which they were related originally. On the other hand, the operations have themselves properties that can be described and

[9] See Ruiz-Zúñiga 1988 (b).

[10] An interesting paper on the ontology of mathematics is done in Parsons 1971.

[11] This would be a very useful tool in the consideration of certain problems of Mathematics Education. There are philosophical issues behind many of the problems of current Mathematics Education; see Grattan-Guinness 1973, and D'Ambrosio 1985, for interesting approaches.

[12] My approach is empiricist. For an interesting recent description of Empiricism in the Philosophy of Mathematics, see Barabashev 1988.

[13] See Bell 1940, Bourbaki 1974 and Boyer 1968.

[14] The reasons why this is possible can be described in different ways. See Kitcher 1983, 1988.

studied. Mathematics operations form abstract structures. These structures can be applied to different objects. What I want to say is that certain structures can be used to explain certain processes of the world. When the structure is applied the system which is created not necessarily corresponds or describes the objects reality. It is possible that another structure can play that role, or that the objects need to be conceptualized in another way. It may happen that a new structure needs to be created and it is not at the moment available in the Mathematics "store". Thus, the world of the operations of Mathematics has its own rules.

Is the real nature of Mathematics its methods and operations? I would say yes and no. "Yes" because its methods and operations constitute a very important part of Mathematics. The revolution of the Mathematics in the XIX th century provoked this dimension to be widened considerably. Furthermore, this revolution created the possibility of a practice in Mathematics able to focus only on the operations.

And, I would say "no" because the Mathematics objects are still a defining base of Mathematics. There are epistemological reasons behind my former statement. The mental actions involved in Mathematics are not only operations but representations, abstraction performed on objects and concepts or even operations. Mathematics is the result of different types of mental actions. To purge Mathematics of the representations and diverse abstractions and to reduce it to the world of operations is to overview the historical evolution of Mathematics. [15]

The interaction between both worlds (objects and concepts) is in the most intimate sense what Mathematics nature is. Thus, in my approach Mathematics is a combination of objects connected or within the space-time reality and -at the same time- abstract methods and concepts. The predominance of one factor or the other has depended on the precise historical evolution of this science.

Thus, is there truth in Mathematics or not? When we deal only with methods the problem does not arise. When we deal with empirical objects and methods as well, the problem arises, and we should be able to confront our theoretical results with reality just as it happens with the other sciences.

Perhaps, the evolution of science tends increasingly towards this former pattern: a combination of Mathematics methods and objects. The world of objects depends on the empirical actions we humans perform upon reality.

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[15] Kitcher sees this, but he underlines methods and operations instead of objects.

When the history of Mathematics started to emphasize methods without objects it created not only a new type of Mathematics practice but a qualitatively different evolution of Mathematics. [16] The emphasis was no more on description but on the development of new operations, structures, etc.

As a first formulation, we can say Modern Science became the heir of former Mathematics. [17] If we say there are Pure and Applied Mathematics, Applied Mathematics is part of modern science if we define science as an interest and a "concern" about the world and not only on the operation the mind can perform and its properties. The new field of pure Mathematics -world of operations and non-object abstractions- has its own life and rules but the practice that produces it is not aimed to describe reality.

What are the limits of this theoretical trend born from classical Mathematics?. First of all, without the existence of another practice aiming to describe reality it would be just a mentally interesting creation as other expressions of human creativity, and with no utility in the material world. But, because its nature comes from the abstraction of real and physical operations, it could be possible to use in science these results (somehow). If, moreover, there is a "clear" attitude in the scientific community and a sound practice to use Mathematics results upon reality, it can be integrated in the new building of science. Therefore, as a second approach, we can say that Modern Science is a unified building that includes "Pure" Mathematics as a part of its methods, which find a role in an overall aim to theoretically "disentangle" reality. In any approach, what I want to stress is the connection between mathematical methods and concepts and mathematical objects. In my opinion, this connection constitutes -in the long run- the most important motor for the mathematical practice.

By the same token, the methods Mathematics has created have had a relation to the objects where they originated. I mean that the approach to certain objects produces methods that then can be generalized, abstracted, fused with others, etc. The emergence of objects creates the impulse for new operations and new methods. In this sense, I think that the emergence of objects constitutes the long-term creator of Mathematics stages of development. After a new realm is founded (or created), the mathematical practice goes on it and creates a world of results, with or without description of the new object or the process it gave origin to that object. In this sense Mathematics practice and its historical evolution are conditioned by the interplay between the world of objects and the world of concepts and methods which arise from them in the

[16] See Ruiz-Zúñiga 1987 (c).

[17] Cfr. Bell 1940.

long-run. Perhaps not all objects provoke the same results. That is why we can speak about general-objects. But it represents an attempt to describe the development of Mathematics as it has been historically. That is to say that we could attempt to describe the evolution of Mathematics through stages associated with the consideration given to the objects. We can understand the history of Mathematics divided in parts, whose definition comes from the specific apprehension of these objects. This can be done in very wide terms using the general-objects we introduced here, and also using the particular objects. Thus, within this division of the history of Mathematics it is possible to study the methods associated to each object and to describe its limits and properties. Later, we can study the history of the methods and their applications everywhere, and then further abstractions and "deconstruction and reconstruction" of them. These combined studies would give us the description of the evolution of Mathematics.[18]

Thus, the approach I have suggested is ontological and epistemologically "biased", but it is based on an interpretation of the historical development of Mathematics (although, opinionated itself too).

In general terms, we can think about the history of science as the interaction between two arrows. One is the empirical arrow that connects with the physical world in the wide sense, and where sensorial experience is basic (observation, physical contact, experiment, etc.). And the arrow of abstraction and mental operations. Both have been together defining precise stages and phases in the evolution of knowledge. Certain emphasis in the arrow of mental elaboration gave us Mathematics; an emphasis in the empirical arrow gave us what is normally called natural science. In the first stages of science, there was not such a big difference between both fields. The separation began with modernity. That is to say it emerged a new relationship between these two arrows. However, the extraordinary development of conceptualization of the “experimental” sciences has created a "mathematics-ation" of large parts of them, multiplying the links with Mathematics. In other terms, Mathematics plays a wider and increasing role in modern science, and the wider use of Mathematics by a particular science expresses its development in conceptualization and theory-formulation.

The essence of Mathematics is not associated with the experiment and the direct physical contact with the world, but with the mental processes of abstractions and operations upon concepts derived from empirical objects. During ages both processes were done normally by the same practitioners. [19]

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[18] Other historical accounts can be consulted: Kramer 1970, Lakatos 1978, Archibald 1949.

[19] See Bell 1937.

With respect to this, we should understand that the emergence of modern science created different conditions for Mathematics in a special form. The big impulse towards experimentation in science, the rise of the modern abstract Mathematics, and the development of high levels of professional specialization provoked that results of the experimental practice became the main source of objects for Mathematics. In the structure of modern science the role of the apprehension of the objects for Mathematics and a wide range of their conceptualization has been done by scientists who are not mathematicians. It emerged from a separation between practitioners of Mathematics conceptualization and of the following processes of mental abstractions and operations. However, this situation generated that non-mathematicians easily became Mathematics practitioners creating rich new fields and methods in Mathematics. By the same token, many other mathematicians trained in Pure Mathematics have gone into science in order to add vitality to their jobs. It seems clear that without a close and sound connection between mathematicians and other scientists, the development of Mathematics would be diminished in the long run. Without the impulse of new objects extracted from science or in general from the empirical realm, the advancement of Mathematics would become limited and reduced. [20]

When I said before that in "pure" Mathematics there is no truth problem, it should be understood this has epistemological consequences. Are there criteria to decide on the use or performance of the practice? It is obvious that once certain basic rules are accepted, it would be absurd not to follow them. But beyond that point, it seems clear to me that there are no other criteria. Or, perhaps, put in another way, the criteria depend on the acceptance by the community of practitioners. The acceptance by a community of practitioners depends on the criteria they elaborate, and this is historically and socially done. On the other hand, the criteria they elaborate are not arbitrary. There are sets of factors which produce them. Some are related to the same theoretical results (formal consistency, rigor, logical discourse, etc.). There is interplay between these factors that can be studied.

I did not say it before, but the "physical-devoided" operations I have mentioned are connected themselves to the general structure of the being. I mean by that: there is an external world (with laws, etc.) to which we belong. I do not say that these operations are determined by the material world, but that they are possible in a context where both the actors within the "play" of knowledge are material realities.

Finally, one of the consequences of this former premise is related to an answer

[20] This kind of historical studies has not been done very often.

of the following question: assuming the existence of other intelligent beings within the universe, would be a contradiction between our structure of knowledge and theirs? If the structure of our knowledge depends on the external material world, thus a contradiction would emerge only if there are contradictions between our universe and the universe those creatures live in. It is not demonstrated that the universe we use to think about is the only "one". It is possible to think there are different universes with different-to-the-universe-we-are-in physical laws, and incompatible or contradictory to our world. In this hypothetical case, the same concept of "contradiction" would be redefined in response to that physical reality. Then, our current knowledge is understood to be in connection to the being we belong to, and to which we normally affirm we can know. In any case, there would be a kind of contradiction due to the fact they would be different forms of knowledge, because the form and the limits of our knowledge are determined by what we are in physical and in cultural terms. That is to say the dimension of perceptions and the dimension of concepts and methods (which evolves historically and culturally) create our specific form and structure of knowledge, which would make the difference even if the "contradictions" due to universe-differences turn out not to be inescapable. Here, we are stressing again the importance of the dimension of perceptions within our theory of knowledge we have underlined all over this paper.

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